STRUCTURE THEOREMS FOR THE SYMMETRIC GROUPS ACTING ON ITS NATURAL MODULE

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Abstract

This paper gives an explicit structure theorem for the symmetric group acting on the symmetric algebra of its natural module. Let G be the symmetric group on $x_1, \dots, x_n$ and let d_i be the i^{th} elementary symmetric polynomial in the x_i's. We show that if we take monomial representations discussed in [7, Section 3] to be the modules V_I, then we have an isomorphism of kG-modules $k[x_1, \dots, x_n] \cong \bigoplus_{\{n\} \subseteq I \subseteq [n]} k[d_I] \otimes_k V_I$.

This paper gives a structure theorem for the symmetric group, G, acting on its natural module, which gives us a kG-decomposition of the graded components of $S = k[x_1, \dots, x_n]$, where k is a unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$ for $a, b \in k$. Which is to say, for $d_1, \dots, d_n$ the elementary symmetric polynomials in $x_1, \dots, x_n$, we give kG-submodules of S, V_I for $I \subseteq \{1, \dots, n\}$ $n \in I$, such that the multiplication map $\bigoplus_{\{n\} \subseteq I \subseteq \{1,\dots,n\}} k[d_I] \otimes_k V_I \to S$ is a kG-isomorphism.

In fact the monomial representations discussed in [7, Section 3] maybe be taken as the modules, V_I, occurring in a structure theorem. Many of the intermediate steps will be similar to those from [7], but the fact that we get a structure theorem is new as is the observation that we may use e_I, rather than the e'_I used by Kemper. Note that although the ring k need not be commutative, we require that $ax_i = x_i a$ for $i = 1, \dots, n$ and for all $a \in k$.

It will turn out that in this example of a structure theorem all V_I with $n \notin I$ are zero, this was also true for the upper triangular structure theorem.

For more information on structure theorems see [4], [5], [6], [10] and [9]. A more verbose exposition of this material and additional examples of structure theorems can be found in [8].

1 Definition and Results in the Literature

Let k be a unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$, which need not be commutative. Let $R = k[d_1, \dots, d_n]$ be the $\mathbb{N}$ graded polynomial k-algebra in the indeterminants $d_1, \dots, d_n$, with $\deg(d_i) > 0$ but not necessarily

with $\deg(d_1) = 1$. Let G be any finite group and let S be a finitely generated $\mathbb{Z}$-graded RG-module.

Definition 1.1. *With notation as above, a* Structure Theorem *for S over RG is a set of finitely generated kG-submodules, $X_I \subseteq S$, one for each $I \subseteq \{1, \dots, n\}$, such that the map:*

$$\phi : \bigoplus_{I \subseteq \{1,\dots,n\}} k[d_i | i \in I] \otimes_k X_I \to S$$
$$\phi : d \otimes_k x \mapsto dx$$

is an isomorphism of kG-modules.

Note that the map ϕ is split over kG, as it is a kG-isomorphism. As the module being mapped from is not an R-module, it cannot hope to be an R-map, however the following lemma is straightforward.

Lemma 1.2. *For each component of the sum, the map:*

$$\phi_I : k[d_i | i \in I] \otimes_k X_I \to S$$
$$\phi_I : d \otimes_k x \mapsto dx$$

is a $k[d_i | i \in I]G$-homomorphism.

If we insist that k is a field, then we know that a structure theorem exists for the symmetric group acting on its natural module by the following arguments.

Theorem 1.3 (Symonds 2006)**.** *[9] Let k be a field and let $R = k[d_1, \dots, d_n]$ be the graded polynomial ring with $\deg(d_i) > 0$ for all i, let G be a finite group graded in degree 0 and let S be a finitely generated $\mathbb{Z}$-graded RG-module. A structure theorem for S exists exactly when only finitely many isomorphism classes of indecomposable kG-modules occur as summands of S.*

Note that since we insisted that the X_I are finitely generated it is not the case that every S trivially has a structure theorem given by $X_\emptyset = S|_{kG}$.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring in n variables graded in degree 1. With respect to the basis $x_1, \dots, x_n$ of the degree 1 component of S, let P denote a finite subgroup of the upper triangular of matrices with 1's on the diagonal.

Theorem 1.4 (Karagueuzian and Symonds 2007)**.** *[5, Theorem 1.1] For k a finite field, S and P as immediately above and $R \subset S^P$ a particular Noether normalization of S^P, S has a structure theorem over RP.*

Any group acting on S with grading preserving algebra automorphisms is defined by its action on the degree 1 component of S. Let P be any Sylow-p-subgroup of G. It can be shown that we may chose a basis of the degree 1 component of S such that the elements of P are represented by upper triangular matrices with 1's on the diagonal. A similar argument to Theorem 1.4 (found

in the proof of [13, Corollary 4.2]) tells us that S has a structure theorem over RP. Since P is a Sylow-p-subgroup of G, this tells us that S has finitely many isomorphism classes of indecomposable kG-summands. Hence S has a structure theorem over RG. All together this shows:

Corollary 1.5. *Let k be a field of characteristic p. For $S = k[x_1, \dots, x_n]$, with $\deg(x_i) = 1$ for $i = 1, \dots, n$, G a finite group of grading preserving algebra automorphisms and $R \subseteq S^G$ a polynomial ring such that S is a finite R-module, S has a structure theorem over RG. cf. [9, Corollary 1.2].*

2 Notation

We now fix notation which we will use for the rest of the paper.

Take k to be any unital ring such that $ab = 0$ implies $a = 0$ or $b = 0$. Fix an $n \in \mathbb{N}_{>0}$, let $S = k[x_1, \dots, x_n]$ and let $G = \mathrm{Sym}(x_1, \dots, x_n)$ be the symmetric group on the variables $x_1, \dots, x_n$. Let d_i be the i^{th} elementary symmetric polynomial in $x_1, \dots, x_n$ e.g. $d_1 = x_1 + \dots + x_n$, $d_n = x_1 x_2 \dots x_n$ and for $i \in \{1, \dots, n\}$:

$$d_i = \sum_{g \in G/\mathrm{stab}_G(x_1 \dots x_i)} g(x_1 \dots x_i).$$

Let $R = k[d_1, \dots, d_n]$. It is well known that the d_i are algebraically independent (a result sometimes called the fundamental theorem of symmetric polynomials), so R is a polynomial k-algebra.

Note that $\mathrm{stab}_G(x_1 \dots x_i)$ is the stabilizer of the monomial $x_1 \dots x_i$, which is the same as the stabilizer of the set $\{x_1, \dots, x_i\}$. Elements of this group are made up of a permutation of $x_1, \dots, x_i$ and a permutation of $x_{i+1}, \dots, x_n$.

For any $m \in \mathbb{N}$ let $[m] = \{1, 2, \dots, m\}$. For $I = \{i_1, \dots, i_m\} \subseteq [n]$, let $\tilde{d}_I = d_{i_1} \dots d_{i_m} = \prod_{i \in I} d_i$, we let $\tilde{d}_\emptyset = 1$. Let d_I denote the set $\{d_i | i \in I\}$ and $k[d_I]$ denote the polynomial ring $k[d_i | i \in I]$.

Let $\mathrm{lm}_{\mathrm{lex}}$ denote the leading monomial in the usual lexicographical ordering on monomials in $x_1, \dots, x_n$. For $I = \{i_1, \dots i_m\} \subseteq [n]$ with $n \in I$, set $e'_I = \mathrm{lm}_{\mathrm{lex}}(\tilde{d}_I)/d_n$ and let V'_I be the kG-module generated by e'_I. An element of R of the form $d_{i_1}^{t_1} \dots t_{i_m}^{t_m}$ we will call a d_I-monomial, and if $I = [n]$ we may shorten this to a d-monomial. Likewise x_I and x-monomials, are elements of S of the form $x_{i_1}^{t_1} \dots x_{i_m}^{t_m}$, with $i_j \in I$ and $[n]$ respectively.

Note that if we defined $e''_I = \mathrm{lm}_{\mathrm{lex}}(\tilde{d}_I)$ and $V''_I = \langle e''_I \rangle$ for any $I \subseteq [n]$, then for any I with $n \notin I$ we would have $V''_I \cong V''_{I \cup \{n\}}$, and for any I with $n \in I$ we would have $V'_I \cong V''_I$. So no new isomorphism classes of module occur for V''_I with $n \notin I$. In both cases the isomorphism is given by multiplication by d_n.

This notation is summarized in the top part of table below, for now ignore the bottom two rows as G-lm has not yet been defined.

k	unital ring such that $ab = 0 \implies a = 0$ or $b = 0$ for all $a, b \in k$
S	$k[x_1, \ldots, x_n]$
G	$\mathrm{Sym}(x_1, \ldots, x_n)$
d_i	i^{th} elementary symmetric polynomial in $x_1, \ldots, x_n$ i.e. $d_i = \sum_{g \in G/stab_G(x_1 \ldots x_i)} g x_1 \ldots x_i$
R	$k[d_1, \ldots, d_n]$
$[m]$	$\{1, 2, \ldots, m\}$
I	$I \subseteq [n], I = \{i_1, \ldots, i_{\|I\|}\}$
$\tilde{d_I}$	$d_{i_1} \ldots d_{i_{\|I\|}} = \prod_{i \in I} d_i$ for $\{n\} \subseteq I \subseteq [n]$
d_I	$\{d_i \| i \in I\}$
e'_I	$\mathrm{lm}_{\mathrm{lex}}(\tilde{d_I})/d_n$ for $\{n\} \subseteq I \subseteq [n]$
V'_I	the kG-module generated by e'_I for $\{n\} \subseteq I \subseteq [n]$
e_I	an element of S such that $G\text{-lm}(e_I) = \{\mathrm{lm}_{\mathrm{lex}}(\tilde{d_I})/d_n\}$, $\mathrm{stab}_G(e_I) = \mathrm{stab}_G(\mathrm{lm}_{\mathrm{lex}}(e_I))$ for $\{n\} \subseteq I \subseteq [n]$ and the coefficient of the $\succ$-leading monomial is a unit (e.g. $e_I = e'_I$)
V_I	the kG-module generated by e_I for $\{n\} \subseteq I \subseteq [n]$

The result we are aiming for is:

Theorem 2.1. *With notation as above, we have a structure theorem:*

$$S \cong \bigoplus_{\{n\} \subseteq I \subseteq [n]} k[d_I] \otimes_k V'_I$$

Where the map from right to left is the kG-homomorphism $d \otimes_k v \mapsto dv$

This is an immediate corollary of Theorem 5.2, where e'_I and V'_I are replaced by e_I and V_I.

Using e_I, rather than e'_I, does make the notation a little more messy but being able to use e_I allows more flexibility. It may also be useful for considering localizations of S. For example, assume the e_I version of the theorem holds and fix $r \in [n]$, then the following choices for e_I are allowed:

$$e_I = \begin{cases} e'_I & \text{if } r \notin I \\ d_r e'_{I-\{r\}} & \text{if } r \in I \end{cases}$$

If $r \notin I$ then $d_r V_I \subseteq \mathrm{Im}(1_R \otimes_k V_{I \cup \{r\}})$. On the other hand if $r \in I$, since the theorem holds, we have $d_r V_I = \mathrm{Im}(d_r \otimes_k V_I)$. So for all $I \subseteq [n]$ with $n \in I$, we have $d_r V_I \subset \mathrm{Im}\left(k[d_{I \cup \{r\}}] \otimes_k V_{I \cup \{r\}}\right)$. This tells you that for S_{d_r}, the localization of S by d_r, we have a split isomorphism of kG-modules:

$$S_{d_r} \cong \bigoplus_{\{I \subseteq [n] | r, n \in I\}} k[d_I][d_r^{-1}] \otimes_k V_I$$

where the isomorphism from right to left is given by multiplication.

The two main tools we use are the $\succ$-leading monomials and the *reduced form.*

3 Leading Monomials

The following definitions are similar to [7, Section 3 Definition 13].

Definition 3.1 ($\succ, \succcurlyeq, \approx, M(-), G\text{-lm}$)**.** *For two x-monomials, $y, z \in S$, pick $g, h \in G$ such that $gy \geq_{\text{lex}} g'y$ for all $g' \in G$ and $hz \geq_{\text{lex}} h'z$ for all $h' \in G$, we say that $y \succcurlyeq z$ if $gy \geq_{\text{lex}} hz$, otherwise $y \prec z$.*

We say that $x \approx y$ if $x \preccurlyeq y$ and $y \preccurlyeq x$, i.e. if there exists $g, h \in G$ such that $gx = hy$.

For $u \in S$, define $M(u)$ to be the set of x-monomials occurring in u (with non-zero coefficient) and define

$$G\text{-lm}(u) := \{x \in M(u) | x \succcurlyeq y \text{ for all } y \in M(u)\}$$

For a set X such that $x \approx y$ for all $x, y \in X$, write $X \approx m$ if $\forall x \in X, x \approx m$. Note that $\forall x, y \in G\text{-lm}(u), x \approx y$, so $G\text{-lm}(u) \approx m$ makes sense.

Note the distinction between $G\text{-lm}(u) \approx m$ and $G\text{-lm}(u) = \{m\}$ for $u, m \in S$, m an x-monomial. The former says that the leading monomials of u in the $\succcurlyeq$ ordering are all equal to gm for some $g \in G$. The latter says that there is exactly one $n \in M(u)$ which is maximal in the $\succcurlyeq$ ordering and this n is equal to m.

Let e_I and V_I be as in the box from Section 2, i.e. for $I \subseteq [n]$ with $n \in I$: e_I is an element of S such that $G\text{-lm}(e_I) = \{\text{lm}_{\text{lex}}(\tilde{d_I})/d_n\} = \{e'_I\}$, $\text{stab}_G(e_I) = \text{stab}_G(e'_I)$ and the coefficient of the $\succ$-leading monomial is a unit; V_I is the module kGe_I.

The condition that $G\text{-lm}(e_I) = \{e'_I\}$ could be relaxed to $G\text{-lm}(e_I) = \{g \cdot e'_I\}$, or we could say $G\text{-lm}(e_I) \approx \{e'_I\}$ and $|G\text{-lm}(e_I)| = 1$. We gain no benefit from this as the next lemma tells us that for such an e_I we would have $G\text{-lm}(g^{-1}e_I) = \{e'_I\}$, so the V_I obtained in this way are the same. So we insist that $G\text{-lm}(e_I) = \{e'_I\}$.

Lemma 3.2. *Let d be a d-monomial considered as an element of S and u, v, w be any elements of S then:*

1. $\text{lm}_{\text{lex}}(uv) = \text{lm}_{\text{lex}}(u)\text{lm}_{\text{lex}}(v)$
2. $G\text{-lm}(d) \approx \text{lm}_{\text{lex}}(d)$
3. $G\text{-lm}(d) = \{g \cdot \text{lm}_{\text{lex}}(d) | g \in G\}$
4. *For any $g \in G$ we have $G\text{-lm}(u) \approx G\text{-lm}(gu)$.*
5. *Let m be an x-monomial with $m \in G\text{-lm}(u)$, $G\text{-lm}(u) = Gm \cap M(u)$*
6. *For $G\text{-lm}(u)$ and $G\text{-lm}(v)$ disjoint, $G\text{-lm}(u + v) \subseteq G\text{-lm}(u) \cup G\text{-lm}(v)$, in particular $G\text{-lm}(u + v) \approx G\text{-lm}(u)$ or $G\text{-lm}(u + v) \approx G\text{-lm}(v)$.*
7. $G\text{-lm}(gu) = gG\text{-lm}(u)$ *for all $g \in G$.*

Proof. (1) follows from $a \geq_{\text{lex}} b \implies ac \geq_{\text{lex}} bc$ for x-monomials a, b, c.

(2) and (3) are because d is a d-monomial.

(4) is because $m \approx gm$ and $m \succ n \iff gm \succ gn$, for x-monomial m, n.

(5) if $n \in G\text{-lm}(u)$, then $n \not\succcurlyeq m'$ for all $m' \in M(u)$, so in particular $n \not\succcurlyeq m$. We already know $m \not\succcurlyeq n$, so $m \approx n$, i.e. $n \in Gm$. Clearly $G\text{-lm}(u) \subseteq M(u)$, so $G\text{-lm}(x) \subseteq Gm \cap M(x)$.

Conversely, if $n \in Gm \cap M(u)$, then $n \approx m$ and $m \succeq m'$ for all $m' \in M(u)$. So $n \succeq m'$, for all $m' \in M(u)$, and $n \in M(u)$, so $n \in G\text{-lm}(u)$.

(6) for $m \in G\text{-lm}(u)$, $n \in G\text{-lm}(v)$, without loss of generality let $m \not\succcurlyeq n$. Then $m \in M(u+v)$, as $m \notin M(v)$, and $m \not\succcurlyeq m'$ for all $m' \in M(u+v)$.

(7) suppose $m \in G\text{-lm}(u)$, then $gm \in Gm \cap M(gu)$ so $gm \in G\text{-lm}(gu)$ by part (5). This shows that $gG\text{-lm}(u) \subseteq G\text{-lm}(gu)$ and $g^{-1}G\text{-lm}(gu) \subseteq G\text{-lm}(g^{-1}gu)$. □

Lemma 3.3. *For e_I and V_I as in the box and $u \in V_I$ with $u \neq 0$, we have $G\text{-lm}(u) \approx e'_I$*

Proof. As $V_I = kGe_I$, for T a transversal of $\text{stab}_G(e_I)$ in G, any non-zero element of V_I may be expressed uniquely as a sum:

$$\sum_{g \in T} \lambda_g g e_I,$$

for $\lambda_g \in k$, with at least one λ_g non-zero.

Since $\text{stab}_G(e_I) = \text{stab}_G(e'_I)$, the T we chose above is a transversal of $\text{stab}_G(e'_I)$ in G. By definition $G\text{-lm}(e_I) = \{e'_I\}$, hence by Lemma 3.2(7), $G\text{-lm}(ge_I) = gG\text{-lm}(e_I) = \{g \cdot e'_I\}$. So for $g, h \in T$ we have that $G\text{-lm}(ge_I)$ and $G\text{-lm}(he_I)$ are disjoint when $g \neq h$.

By repeated application of Lemma 3.2 (6), for $\lambda_g \in k$ with at least one of the $\lambda_g \neq 0$ we have:

$$G\text{-lm}\left(\sum_{g \in T} \lambda_g g e_I\right) \approx G\text{-lm}(g' e_I),$$

for some $g' \in T$ and by Lemma 3.2(4), $G\text{-lm}(g' e_I) \approx e'_I$ for all $g' \in T$. □

Lemma 3.4. *For e_I and V_I as in the box, d a d_I-monomial and $u \in S$, if $G\text{-lm}(u) \approx e'_I$, then $G\text{-lm}(du) \approx \text{lm}_{\text{lex}}(d)e'_I$.*

In particular, for $u \in V_I - \{0\}$ we have: $G\text{-lm}(du) \approx \text{lm}_{\text{lex}}(d)e'_I$.

Proof. Take $m \in G\text{-lm}(u)$, there exists a $g \in G$ such that $gm = e'_I$. By Lemma 3.2(7) $gG\text{-lm}(u) = G\text{-lm}(gu)$, and by Lemma 3.2(4), $G\text{-lm}(gu) \approx G\text{-lm}(u)$. So we may assume $e'_I \in G\text{-lm}(u)$ and $\text{lm}_{\text{lex}}(u) = e'_I$. Hence $\text{lm}_{\text{lex}}(d)e'_I = \text{lm}_{\text{lex}}(du)$ by Lemma 3.2(1), in particular $\text{lm}_{\text{lex}}(d)e'_I \in M(du)$. So it is sufficient to show that $\text{lm}_{\text{lex}}(d)e'_I \succeq n$ for all $n \in M(du)$.

For $d = d_1^{t_1} \dots d_n^{t_n}$:

$$M(du) = \left\{ \left(\prod_{i=1}^{n} \prod_{j=1}^{t_i} g_{i,j} \mathrm{lm}_{\mathrm{lex}}(d_i) \right) a \,\middle|\, a \in M(u), g_{i,j} \in G \right\}$$

That $G\text{-lm}(u) \approx e'_I$, implies that for all $h \in G$ and all $a \in M(u)$, we have $e'_I \geq_{\mathrm{lex}} ha$. Clearly $\mathrm{lm}_{\mathrm{lex}}(d_i) \geq_{\mathrm{lex}} g_{i,j}\mathrm{lm}_{\mathrm{lex}}(d_i)$, so $\mathrm{lm}_{\mathrm{lex}}(d)e'_I \geq_{\mathrm{lex}} hn$ for all $n \in M(du)$.

The "in particular" statement follows from Lemma 3.3. $\square$

4 Reduced Form

The following definition is equivalent to [7, Section 3 Definition 10], where it is described as a generalization of Göbel's concept of "special" terms.

Definition 4.1. *For an x-monomial $m \in S, m = x_1^{r_1} \dots x_n^{r_n}$, the* reduced form *of m, $\mathrm{Red}(m)$, is the x-monomial $x_1^{r'_1} \dots x_n^{r'_n}$ where:*

- $|\{r'_i | i = 1, \dots, n\}| = |\{r_i | i = 1, \dots, n\}| = a \leq n$
- $\{r'_i | i = 1, \dots, n\} = \{0, 1, \dots, a-1\}$
- $r'_i < r'_j \iff r_i < r_j$ *for all i, j.*

We say that an x-monomial, m, is in reduced form *if $m = \mathrm{Red}(m)$.*

Note that for every x-monomial in S, m, there exists a $g \in G$ such that gm is the leading x-monomial of some d-monomial. This is simply the observation that every x-monomial $m = x_1^{m_1} \dots x_n^{m_n}$ with $m_1 \geq m_2 \geq \cdots \geq m_n$ can be written as $x_1^{a_1}(x_1x_2)^{a_2}(x_1x_2x_3)^{a_3} \dots (x_1 \dots x_n)^{a_n}$. The idea of this definition is that the reduced form of m tells us which d_i occur at least once in this d-monomial by looking at when the powers change. For example: the reduced form of $x_1^4x_2^4x_3$ is $x_1^2x_2^2x_3$ and this is the leading monomial of d_2d_3. Another example is $\mathrm{Red}(x_2^2x_3^3) = x_2x_3^2$, which the group element (x_1, x_3) applied to the leading monomial of d_1d_2.

We show, in Corollary 4.8, that one way to think of $\mathrm{Red}(d)$, for d a d-monomial, is to write out the product of the leading monomials of the d_i's vertically, then get rid of the repetitions and the d_n's. For example, let $I =$

$\{i_1, \dots i_a\}, i_1 < i_2 < \dots < i_a = n$ and $d = d_{i_1}^{t_1} \dots d_{i_a}^{t_a}$ we may write $\mathrm{lm}_{\mathrm{lex}}(d)$ as:

$$\begin{array}{rl}
\mathrm{lm}_{\mathrm{lex}}(d_{i_1})^{t_1} & \left\{ \begin{array}{l} x_1 \dots x_{i_1} \\ x_1 \dots x_{i_1} \end{array} \right. \\
\mathrm{lm}_{\mathrm{lex}}(d_{i_2})^{t_2} & \left\{ \begin{array}{l} x_1 \dots x_{i_1} \dots x_{i_2} \\ x_1 \dots x_{i_1} \dots x_{i_2} \end{array} \right. \\
\vdots & \left\{ \begin{array}{l} x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_j} \\ x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_j} \end{array} \right. \\
\mathrm{lm}_{\mathrm{lex}}(d_{i_{a-1}})^{t_{a-1}} & \left\{ \begin{array}{l} x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \dots x_{i_{a-1}} \\ x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \dots x_{i_{a-1}} \end{array} \right. \\
\mathrm{lm}_{\mathrm{lex}}(d_n)^{t_n} & \left\{ \begin{array}{l} x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \dots x_{i_{a-1}} \dots x_n \\ x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \dots x_{i_{a-1}} \dots x_n. \end{array} \right.
\end{array}$$

So the reduced form is just:

$$\begin{array}{l}
x_1 \dots x_{i_1} \\
x_1 \dots x_{i_1} \dots x_{i_2} \\
x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \\
x_1 \dots x_{i_1} \dots x_{i_2} \dots x_{i_*} \dots x_{i_{a-1}},
\end{array}$$

which is clearly $\mathrm{lm}_{\mathrm{lex}}(d_{i_1} \dots d_{i_{a-1}})$.

Lemma 4.2. *For x-monomials $m = x_1^{m_1} \dots x_n^{m_n}$ and $r = x_1^{r_1} \dots x_n^{r_n}$, $\mathrm{Red}(m) = \mathrm{Red}(r)$ if and only if we have $(m_i > m_j) \iff (r_i > r_j)$.*

Proof. Let $\mathrm{Red}(m) = x_1^{m'_1} \dots x_n^{m'_n}$ and $\mathrm{Red}(r) = x_1^{r'_1} \dots x_n^{r'_n}$. If $\mathrm{Red}(m) = \mathrm{Red}(r)$ then $m'_i = r'_i$ and $(m_i > m_j) \iff (m'_i > m'_j) \iff (r'_i > r'_j) \iff (r_i > r_j)$.

For the converse: if $(m_i > m_j) \iff (r_i > r_j)$, then $(m'_i > m'_j) \iff (r'_i > r'_j)$, and the longest increasing chain of m'_i is the same length as the longest increasing chain of r'_i. Hence $\{r'_i | i = 1, \dots, n\} = \{m'_i | i = 1, \dots, n\}$. □

Lemma 4.3. *For an x-monomial, m, $\mathrm{Red}(m) \approx e'_I$ for some $I \subseteq [n]$, $n \in I$.*

Proof. This follows directly from the definition. Let m' be a monomial in reduced form, $m' = x_1^{m'_1} \dots x_n^{m'_n}$ and $\{m'_1, \dots, m'_n\} = \{0, \dots, a\}$. Then there exists a $g \in G$ such that $gm' = m'' = x_1^{m''_1} \dots x_n^{m''_n}$ with $m''_1 \geq m''_2 \geq \dots \geq m''_n = 0$. We may write:

$$m'' = (x_1 \dots x_{i_1})^a (x_{1+i_2} \dots x_{i_2})^{a-1} \dots (x_{1+i_{a-1}} \dots x_{i_a})^1 (x_{1+i_a} \dots x_n)^0.$$

But this is equal to: $\mathrm{lm}_{\mathrm{lex}}(d_{i_1})\mathrm{lm}_{\mathrm{lex}}(d_{i_2}) \dots \mathrm{lm}_{\mathrm{lex}}(d_{i_a})$, and so: $m' \approx m'' = e'_{\{i_1, \dots, i_a, n\}}$. □

Lemma 4.4. *For x-monomials $x, y \in S$: $\mathrm{Red}(gx) = g\mathrm{Red}(x)$ and $x \approx y$ implies $\mathrm{Red}(x) \approx \mathrm{Red}(y)$. cf. [7, Section 3 Lemma 12].*

Proof. We first show that $\mathrm{Red}(g^{-1}x) = g^{-1}\mathrm{Red}(x)$ for any $g \in G$, this of course shows that $\mathrm{Red}(gx) = g\mathrm{Red}(x)$. Let $x = x_1^{r_1} \dots x_n^{r_n}$ and $\mathrm{Red}(x) = x_1^{r'_1} \dots x_n^{r'_n}$. For $\mathrm{Red}(g^{-1}x) = x_1^{s_1} \dots x_n^{s_n}$ and $g^{-1}\mathrm{Red}(x) = x_1^{t_1} \dots x_n^{t_n}$, we must have that $\{r'_i | i = 1, \dots, n\} = \{s_i | i = 1, \dots, n\} = \{t_i | i = 1, \dots, n\} = \{0, 1, \dots, a-1\}$, so by Lemma 4.2 it remains to show that $s_i > s_j$ if and only if $t_i > t_j$.

We defined G as acting on $\{x_1, \dots, x_n\}$, this gives us an action on $\{1, \dots, n\}$ via $gx_i = x_{g(i)}$. In this notation $g^{-1}(x_1^{r_1} \dots x_n^{r_n}) = x_1^{r_{g(1)}} \dots x_n^{r_{g(n)}}$. Hence $t_i = r'_{g(i)}$, so $t_i > t_j$ if and only if $r'_{g(i)} > r'_{g(j)}$ and by the definition of $\mathrm{Red}(x)$ this is if and only if $r_{g(i)} > r_{g(j)}$. Likewise the definition of $\mathrm{Red}(g^{-1}x)$ states that $s_i > s_j$ if and only if $r_{g(i)} > r_{g(j)}$. Hence $\mathrm{Red}(gx) = g\mathrm{Red}(x)$ by Lemma 4.2.

To show that $x \approx y$ implies $\mathrm{Red}(x) \approx \mathrm{Red}(y)$, note that if $gx = hy$ then $\mathrm{Red}(gx) = \mathrm{Red}(hy)$. So by the above, $g\mathrm{Red}(x) = h\mathrm{Red}(y)$, which is the same as saying $\mathrm{Red}(x) \approx \mathrm{Red}(y)$. □

Definition 4.5. *If X is a set of x-monomial such that $x \approx y$ for all $x, y \in X$ (e.g. $X = G\text{-}\mathrm{lm}(u)$), then by $\mathrm{Red}(X)$ we mean $\{\mathrm{Red}(x) | x \in X\}$.*

Note that by Lemma 4.4, if $\forall x, y \in X, x \approx y$ then $\forall x', y' \in \mathrm{Red}(X), x' \approx y'$, so it makes sense to talk about $\mathrm{Red}(X) \approx m$ when $x \approx y$ for all $x, y \in X$.

Lemma 4.6. *Let e'_I be as in the box and let $u \in S$ be such that $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(u)) = e'_I$. Then for all $t \in I$ we have $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(d_t u)) = e'_I$.*

Proof. Let $m = \mathrm{lm}_{\mathrm{lex}}(u)$ with $m = x_1^{m_1} \dots x_n^{m_n}$ and $e'_I = x_1^{r_1} \dots x_n^{r_n}$. $\mathrm{Red}(m) = e'_I$ implies $m_i > m_j \iff r_i > r_j$ by Lemma 4.2.

For $I = \{i_1, \dots, i_a, n\}$ with $1 \le i_1 < i_2 < \dots < i_a < n$, by definition we have $e'_I = \mathrm{lm}_{\mathrm{lex}}(\prod_{j=1}^a d_{i_j})$. By Lemma 3.2(1) this is equal to $(x_1 \dots x_{i_1})(x_1 \dots x_{i_2}) \dots (x_1 \dots x_{i_{a-1}})$. Collecting all the powers of x_i together we get

$$e'_I = (x_1 \dots x_{i_1})^a (x_{i_1+1} \dots x_{i_2})^{a-2} \dots (x_{i_{a-2}+1} \dots x_{i_a})^1 \tag{4.7}$$

So for $i, j \in [n]$ with $i > j$, we have $r_i > r_j$ if and only if $\exists l \in I$ such that $i \ge l > j$. Hence $m_i > m_j$ if and only if $\exists l \in I$ such that $i \ge l > j$.

By Lemma 3.2(1) $\mathrm{lm}_{\mathrm{lex}}(d_t u) = \mathrm{lm}_{\mathrm{lex}}(d_t)\mathrm{lm}_{\mathrm{lex}}(u) = \mathrm{lm}_{\mathrm{lex}}(d_t)m$. Let $\mathrm{lm}_{\mathrm{lex}}(d_t u) = x_1^{m'_1} \dots x_n^{m'_n}$, then:

$$x_1^{m'_1} \dots x_n^{m'_n} = \mathrm{lm}_{\mathrm{lex}}(d_t u) = (x_1 \dots x_t)m = x_1^{1+m_1} \dots x_t^{1+m_t} x_{t+1}^{m_{t+1}} \dots x_n^{m_n}.$$

We now compare (m_i, m_j) and (m'_i, m'_j) for any pair of $i, j \in [n]$.

For $i, j \le t$: we have $m'_i = m_i + 1$ and $m'_j = m_j + 1$, so we have $(m'_i > m'_j) \iff (m_i > m_j)$.

For $t < i, j$: likewise we have $m'_i = m_i$ and $m'_j = m_j$, so we have $(m'_i > m'_j) \iff (m_i > m_j)$.

For $i \le t < j$: we have $m'_i = m_i + 1$ and $m'_j = m_j$, so $m'_i > m'_j$. But, by the observation following Equation 4.7 and the fact that $t \in I$, we also have $m_i > m_j$.

Hence by Lemma 4.2, $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(d_t u)) = \mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(u)) = e'_I$. □

Corollary 4.8. *For $\{n\} \subseteq I \subseteq [n]$, e_I and e'_I as defined in the box and d a d_I-monomial:* $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(d)e'_I) = e'_I$.

Proof. By Lemma 4.6 $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(d_t e_I)) = e'_I$ for every $t \in I$. So by repeated application of this lemma for any d_I-monomial, d, $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(de_I)) = e'_I$. By Lemma 3.2(1) $\mathrm{lm}_{\mathrm{lex}}(de_I) = \mathrm{lm}_{\mathrm{lex}}(d)e'_I$. □

5 Main Theorem

We now draw together the results of the previous sections to prove that we have a structure theorem.

Lemma 5.1. *For e_I,e'_I and V_I as in the box, given distinct d_I-monomials $r, r_1, \ldots, r_m$, we have $rV_I \cap (\sum_{i=1}^m r_i V_I) = \{0\}$. In particular we have $rV_I \cap r'V_I = 0$ for d_I-monomials $r \neq r'$.*

For $u \in V_I - \{0\}$ and $d \in k[d_I]$ we have $\mathrm{Red}(G\text{-}\mathrm{lm}(du)) \approx e'_I$.

Conversely, if m is an x-monomial then there exists an $I \subseteq [n]$ with $n \in I$, a d_I-monomial, r, and a $g \in G$ such that $G\text{-}\mathrm{lm}(rge_I) = \{m\}$

Proof. In this proof we show that the result holds for a d_I-monomial, then use Lemma 3.2(6) to get the result about an arbitrary element of $k[d_I]$.

First we make a general observation. By Lemma 3.4 for r any d_I-monomial and any $u, v \in V_I - \{0\}$, we have $G\text{-}\mathrm{lm}(ru) \approx G\text{-}\mathrm{lm}(rv) \approx \mathrm{lm}_{\mathrm{lex}}(r)e'_I$. For r' a d_I-monomial $r \neq r'$, we have $\mathrm{lm}_{\mathrm{lex}}(r)e'_I \neq \mathrm{lm}_{\mathrm{lex}}(r')e'_I$. Hence $G\text{-}\mathrm{lm}(ru) \not\approx G\text{-}\mathrm{lm}(r'v)$.

To prove that $rV_I \cap (\sum_{i=1}^m r_i V_I) = \{0\}$, let $r, r_1, \ldots, r_m \in R$ be distinct d_I-monomials and $u, u_1, \ldots, u_m$ be non-zero elements of V_I. Then by the above observation, $G\text{-}\mathrm{lm}(r_i u_i) \not\approx G\text{-}\mathrm{lm}(r_j u_j)$ for any $i, j \in [m]$ with $i \neq j$. So by repeated application of Lemma 3.2(6), $G\text{-}\mathrm{lm}(\sum_{i=1}^m r_i u_i) \approx G\text{-}\mathrm{lm}(r_j u_j)$ for some $j \in [m]$. Hence, by the above observation, $G\text{-}\mathrm{lm}(ru) \not\approx G\text{-}\mathrm{lm}(r_j u_j)$ as $r \neq r_j$, so $G\text{-}\mathrm{lm}(ru) \not\approx G\text{-}\mathrm{lm}(\sum_{i=1}^m r_i u_i)$. So $rV_I \cap (\sum_{i=1}^m r_i V_I) = \{0\}$. This proves the first statement, the "in particular" statement follows as a special case or from the observation at the start of the proof.

Now we prove that for $d \in k[d_I]$ we have $\mathrm{Red}(G\text{-}\mathrm{lm}(du)) \approx e'_I$. For r a d_I-monomial, by Lemma 3.4, $G\text{-}\mathrm{lm}(ru) \approx \mathrm{lm}_{\mathrm{lex}}(r)e'_I$. By Corollary 4.8, $\mathrm{Red}(\mathrm{lm}_{\mathrm{lex}}(r)e'_I) = e'_I$. So by Lemma 4.4, $\mathrm{Red}(G\text{-}\mathrm{lm}(ru)) \approx e'_I$. This deals with the case when $d = r$ is a d_I-monomial.

For $d = \sum_{i=1}^m \lambda_i r_i$, with $\lambda_i \in k - \{0\}$ and r_i d_I-monomials, the $G\text{-}\mathrm{lm}(\lambda r_i u)$ are pairwise disjoint. Hence by Lemma 3.2(6), there exists an $j \in \{1, \ldots, m\}$ such that $G\text{-}\mathrm{lm}(du) \approx G\text{-}\mathrm{lm}(r_j u)$. So may prove that $\mathrm{Red}(G\text{-}\mathrm{lm}(du)) \approx e'_I$ using the "d is a d_I-monomial case" proved above.

For the converse: By Lemma 3.2(7), it is sufficient to find I, r and g_m for m with the property that $m \geq_{\mathrm{lex}} gm$ for all $g \in G$. So for the rest of the proof we assume that m has this property.

Now $m = \prod_{i=1}^n \mathrm{lm}_{\mathrm{lex}}(d_i)^{t_i}$, for some $t_i \in \mathbb{N}$. Let $I = \{i | t_i \neq 0\} \cup \{n\}$, so that $m = d_n^{t_n} \prod_{i \in I} \mathrm{lm}_{\mathrm{lex}}(d_i)^{t_i}$. Then e_I divides m and $m = \mathrm{lm}_{\mathrm{lex}}(e_I d_n^{t_n} \prod_{i \in I} \mathrm{lm}_{\mathrm{lex}}(d_i)^{t_i - 1})$. Let $r = d_n^{t_n} \prod_{i \in I} d_i^{t_i - 1}$ then by Lemma 3.4 $G\text{-}\mathrm{lm}(e_I r) = \{m\}$. □

Theorem 5.2. *Let e_I be elements of S such G-$\mathrm{lm}(e_I) = \{\mathrm{lm}_{\mathrm{lex}}(\tilde{d_I})/d_n\}$ and* $\mathrm{stab}_G(e_I) = \mathrm{stab}_G(\mathrm{lm}_{\mathrm{lex}}(e_I))$*. Let V_I be the kG-module generated by e_I. Then as kG-modules we have:*

$$S \cong \bigoplus_{\{n\}\subseteq I\subseteq[n]} k[d_I] \otimes_k V_I$$

is a structure theorem for S, i.e. the map from right to left is the kG-homomorphism $d \otimes_k v \mapsto dv$.

Proof. It is clear that the map is a kG-map as inclusion and multiplication by d_i are kG-maps. It remains to show that the map is a bijection.

Injection: By induction on subsets of $[n]$ containing n. The base case is just the observation that for every $I \subseteq [n]$ with $n \in I$, the map $k[d_I] \otimes_k V_I \to S$ is injective. To see this suppose that $u = \sum_{i=1}^m r_i \otimes u_i \mapsto 0$, where $r_i \neq r_j$ for $i \neq j$ and $u_i \in V_I - \{0\}$. Then $\sum_{i=1}^m r_i u_i = 0$. So by the first statement of Lemma 5.1, $r_1 u_1 = \sum_{i=2}^m r_i u_i = 0$ and thus $u = 0$. So $k[d_I] \otimes_k V_I \to S$ is an injective map.

For the inductive hypothesis, suppose that given, A, a set of subsets of $[n]$ all of which contain n (i.e. $A \subset \mathbb{P}([n])$ and $\forall I \in A, n \in I$), the map $\bigoplus_{I\in A} k[d_I] \otimes_k V_I \to S$ is injective. We want to show that $\left(\bigoplus_{I\in A} k[d_I] \otimes_k V_I\right) \oplus (k[d_J] \otimes_k V_J) \to S$ is injective for $J \subseteq [n]$, $n \in J$ and $J \notin A$.

By Lemma 5.1 for all $v \in \phi(k[d_J] \otimes_k V_J), \mathrm{Red}(G\text{-}\mathrm{lm}(\phi(u))) \approx e'_J$. So it is sufficient to show that for $u = \sum_{I\in A} u_I$ with $u_I \in k[d_I] \otimes_k V_I$, $\mathrm{Red}(G\text{-}\mathrm{lm}(\phi(u))) \not\approx e'_J$.

By Lemma 5.1 $\mathrm{Red}(G\text{-}\mathrm{lm}(\phi(u_I))) \approx e'_I$, it is clear that $e'_I \not\approx e'_{I'}$ for $I \neq I'$. Hence by Lemma 4.4 the $G\text{-}\mathrm{lm}(\phi(u_I))$ are disjoint. By Lemma 3.2(6), $G\text{-}\mathrm{lm}(\sum_{I\in A} \phi(u_I)) \approx G\text{-}\mathrm{lm}(\phi(u_{I_0}))$ for some $I_0 \in A$. By Lemma 5.1 again, $\mathrm{Red}(G\text{-}\mathrm{lm}(\phi(u))) \approx \mathrm{Red}(G\text{-}\mathrm{lm}(\phi(u_{I_0}))) \approx e'_{I_0}$, and $e'_{I_0} \not\approx e'_J$ as $I_0 \neq J$. This shows that the map is in injection.

Surjection: To show that the map is surjective we argue by induction on G-lm, where $G\text{-}\mathrm{lm}(u) > G\text{-}\mathrm{lm}(v)$ if $G\text{-}\mathrm{lm}(u) \succ G\text{-}\mathrm{lm}(v)$ or if $G\text{-}\mathrm{lm}(u) \approx G\text{-}\mathrm{lm}(v)$ and $G\text{-}\mathrm{lm}(u) \supsetneq G\text{-}\mathrm{lm}(v)$.

The least $G\text{-}\mathrm{lm}(u)$ is $\{0\}$, which is clearly mapped onto. For $u \in S$ assume every $v \in S$ with $G\text{-}\mathrm{lm}(v) < G\text{-}\mathrm{lm}(u)$ is mapped onto. Pick $m \in G\text{-}\mathrm{lm}(u)$, $\mathrm{Red}(m) = g \cdot e'_I$ by Lemma 4.3. Then by Lemma 5.1, $\exists r \in k[d_I]$ s.t. $G\text{-}\mathrm{lm}(rge_I) = \{m\}$, hence $G\text{-}\mathrm{lm}(u) > G\text{-}\mathrm{lm}(u - rge_I)$.

$rge_I = \phi(r \otimes_k ge_I)$, and by the inductive hypothesis $u - rge_I = \phi(\tilde{u})$ for some $\tilde{u}$, so $\phi(\tilde{u} + r \otimes_k ge_I) = u$. ☐

Note that the modules V_I are not indecomposable. In fact it may be interesting to calculate the vertices of their indecomposable summands as in the example of the upper triangular group, [5, above Corollary 9.5], the modules which occur in the structure theorem, X_J (written as $\bar{X}_J(I)$ for $J \subseteq I \subseteq \{1, \ldots, n\}$ in the notation of that paper), are induced from a subgroup, U_J, which depends on the the set of invariants $\{d_j | j \in J\}$. Be warned that we have adopted different conventions to [5], in particular, for us structure theorems are a sum of $k[d_i | i \in I] \otimes_k X_I$, but in [5] they are a sum of $k[d_i | i \notin I] \otimes_k X_I$.

It is worth noting that if k were a field, in principal, we could have shown the the map in Theorem 5.2 was either injective or surjective and then compared the Hilbert series of the two modules. However this proved somewhat complicated as for $I = \{i_1, \ldots, i_m\}$ with $i_1 < \cdots < i_m = n$, the dimension over k of V_I is $\frac{|G|}{|\mathrm{stab}_G(e_I)|} = \frac{n!}{i_1!(i_2-i_1)!\ldots(n-i_{m-1})!}$.

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